X-641-69-487
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NASA TM X- 63772

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NOVEMBER 1969

GSFC

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M. Bornatici*
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ABSTRACT

It is shown that for a system composed of two identical electron streams in relative motion across a magnetic field, a new low-frequency propagation band occurs for elliptically polarized waves propagating perpendicular to both the magnetic field and stream motion. No low-frequency cutoff exists if the plasma frequency is higher than the electron cyclotron frequency. This "modified extraordinary mode" becomes unstable for wave numbers k greater than some minimum value, $k_{min}$, when the streaming velocity exceeds a certain threshold value. The case of two identical counterstreaming electron-ion plasmas is also discussed.

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CONTENTS

# MODIFIED EXTRAORDINARY MODE IN MAGNETIZED PLASMAS WITH RELATIVE STREAMING

## I. INTRODUCTION

In a system of two interacting electron plasmas, counterstreaming along a uniform magnetic field, the purely transverse linearly polarized mode propagating perpendicular to the magnetic field is expected to possess a low frequency propagation band and to be unstable.[1] On the other hand, in the presence of two beams of ions moving with equal and opposite velocities through a neutralizing background of electrons and across the field, mixed transverse-longitudinal waves, propagating nearly along the direction of streaming, have been predicted to be unstable.[2] Also, a current across a magnetic field can excite unstable waves propagating obliquely to the magnetic field.[3]

In this paper, we consider a symmetrical double beam system, i.e. a system composed of two identical counterstreaming electron beams with a constant and uniform magnetic field perpendicular to the direction of streaming. The characteristic properties of both stable and unstable waves are investigated for the case of propagation perpendicular to both the magnetic field and stream motion. It is found that the ordinary mode is unaffected by relative streaming, while the extraordinary mode is modified in such a way that it possesses a propagation band below the lower hybrid cutoff frequency. Moreover, the modified extraordinary mode is expected to become unstable when the streaming velocity exceeds a certain threshold value. The analysis is extended to the case in which also the ions take part in the streaming motion.

In Section II, the dispersion relation for the modified extraordinary mode is derived. The instability criterion and the growth rate are found in Section III. In Section IV the propagation properties of the stable modified extraordinary waves are investigated. Finally, in Section V, two counterstreaming electron ion plasmas are considered.

## II. DISPERSION RELATION

Let us consider a homogeneous, infinite, fully ionized gas with relative streaming motions among the charged-particle species in the direction perpendicular to that of an external magnetic field, which is taken along the z-axis. Temperature effects and collisions are disregarded and our analysis is based on the magnetohydrodynamic equations for each species and Maxwell's equations. The system is linearized and Fourier transformed in space and time by assuming perturbations propagating perpendicular to both the magnetic field

and streaming motions and varying as $\exp\left[-i(\omega t - kx)\right]$. The dispersion relation for waves of frequency $\omega$ and wave vector k is given by

$$\begin{vmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{vmatrix} = 0 \qquad (1)$$

The matrix elements are defined as

$$A_{11} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2}, \qquad A_{12} = -\frac{1}{\omega} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \left(ku_\alpha + i\Omega_\alpha\right),$$

$$A_{21} = -\frac{1}{\omega} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \left(ku_\alpha - i\Omega_\alpha\right), \qquad A_{22} = 1 - \frac{c^2 k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \left(1 + \frac{k^2 u_\alpha^2}{\omega^2}\right),$$

$$A_{33} = 1 - \frac{c^2 k^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}, \qquad (2)$$

where $\alpha$ refers to the different particle species (electrons or ions) present in the plasma and summations extend over all species. Also $\omega_{p\alpha}$ and $\Omega_\alpha$ are the plasma and cyclotron frequencies respectively containing the algebraic sign of the charge of each species; $\underline{u}_\alpha$ is the non-relativistic dc velocity of the particles in the unperturbed stream and c is the velocity of light.

The general dispersion relation, given by Equation (1), factors into two equations

$$A_{33} = 0, \qquad \text{i.e.,} \qquad k^2 c^2 = \omega^2 - \sum_\alpha \omega_{p\alpha}^2, \qquad (3)$$

and

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 0 \ . \tag{4}$$

Equation (3) refers to the purely transverse linearly polarized mode, whose electric field is along the external magnetic field $\underline{B}_0$. This mode, referred to as the ordinary (O) mode, is unaffected by both $\underline{B}_0$ and streaming motions. Equation (4) yields the dispersion relation for the extraordinary waves in the presence of streaming. These "modified extraordinary waves," hereafter abbreviated as MX, are mixed transverse-longitudinal waves, elliptically polarized in a plane perpendicular to $\underline{B}_0$[4].

In order to be able to investigate the propagation properties of the MX mode in a simple way, we consider a system composed of two identical counterstreaming plasmas, each of density $n_0/2$, moving with equal and opposite dc velocities, $+u$ and $-u$ respectively. By disregarding the motion of ions (infinite ion mass), from Equations (4) and (2), the following dispersion relation is obtained for the MX mode:

$$k^2 c^2 = \frac{\left(\omega^2 - \Omega_e^2\right)\left(\omega^2 - \omega_-^2\right)\left(\omega^2 - \omega_+^2\right)}{\left(\omega^2 - \omega_H^2\right)\left(\omega^2 - \Omega_e^2 + u^2 \omega_{pe}^2/c^2\right)} \ , \tag{5}$$

where $\omega_H \equiv \left(\Omega_e^2 + \omega_{pe}^2\right)^{1/2}$ is the upper hybrid frequency, and

$$\omega_\pm \equiv \left[\left(\frac{\Omega_e}{2}\right)^2 + \omega_{pe}^2\right]^{1/2} \pm \frac{|\Omega_e|}{2} \ . \tag{6}$$

The characteristics of this mode will be examined in the next two sections.

## III. UNSTABLE MX WAVES

In order to investigate the stability properties of the modified extraordinary mode, it is convenient to write Equation (5) as a cubic equation in $\omega^2$:

$$\omega^6 - A\omega^4 + B\omega^2 - C = 0 \tag{7}$$

where

$$A = 2\omega_H^2 + k^2 c^2 > 0\,, \tag{8}$$

$$B = \omega_H^4 + 2k^2 c^2 \Omega_e^2 + k^2 \omega_{pe}^2 \left(c^2 - u^2\right) \simeq \omega_H^4 + k^2 c^2 \left(2\Omega_e^2 + \omega_{pe}^2\right) > 0\,, \tag{9}$$

$$C = \Omega_e^2 \omega_{pe}^4 + k^2 c^2 \omega_H^2 \left(\Omega_e^2 - \frac{u^2}{c^2}\, \omega_{pe}^2\right). \tag{10}$$

It is possible to show, (see also Reference 5) that all three roots for $\omega^2$ are real and positive, corresponding to stability, if

$$\begin{cases} 4B^3 - A^2 B^2 - 18ABC + 27C^2 + 4A^3 C < 0 \\ C > 0\,. \end{cases} \tag{11}$$

On the other hand, for $\omega^2$ to be a negative or complex root, corresponding to an instability, a sufficient condition is

$$\Omega_e^2 \omega_{pe}^4 + k^2 c^2 \omega_H^2 \left(\Omega_e^2 - \frac{u^2}{c^2}\, \omega_{pe}^2\right) < 0 \tag{12}$$

i.e.,

$$k^2 > \frac{\omega_{pe}^2}{\omega_H^2} \, \frac{\Omega_e^2}{u^2 - \dfrac{c^2 \Omega_e^2}{\omega_{pe}^2}}\,. \tag{13}$$

Condition (13) yields the minimum unstable wave number for a given streaming velocity and shows that waves with infinite wave numbers begin to become

unstable when

$$u > c|\Omega_e|/\omega_{pe} . \tag{14}$$

From the dispersion relation given by Equation (7), it is possible to show that for waves with $k \to \infty$, the growth rate is

$$\gamma = \left[\frac{u^2 \omega_{pe}^2}{c^2} - \Omega_e^2\right]^{1/2} , \tag{15}$$

which is also the maximum growth rate, corresponding to an unstable wave with purely imaginary frequency.

It is useful to derive the equations relating the phases and magnitudes of the electric field components and to examine the polarization for the MX mode. To this end the wave equation for the electric field $\underline{E}$ can be written as $\underline{\underline{A}} \cdot \underline{E} = 0$, where $\underline{\underline{A}}$ is the dispersion matrix defined in Equation (1). For MX waves then, the electric field components are related by two equations ($E_z = 0$)

$$\begin{cases} A_{11} E_x + A_{12} E_y = 0 \\ A_{21} E_x + A_{22} E_y = 0 \end{cases} \tag{16}$$

where the coefficients $A_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) are given by Equations (2), and $E_x$ and $E_y$ are the electric field components parallel and perpendicular to the wave vector $\underline{k}$, respectively. For the system of two identical counterstreaming electronic plasmas, from Equations (16) and (2), it results

$$\frac{E_x}{E_y} = -i \frac{|\Omega_e|}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \omega_H^2} , \tag{17}$$

with $\omega$ being the solution of the dispersion relation (5). It appears, therefore, that the MX mode is in general a mixed transverse-longitudinal mode, elliptically polarized in the plane containing the wave vector $\underline{k}$ and the streaming velocity $\underline{u}$.

However, it is possible to show that for stable MX waves with very large wave numbers $|E_x| > |E_y|$, i.e., the longitudinal component of the electric field is greater than the transverse component.[6] On the other hand, $|E_x| < |E_y|$ when $u > 2^{1/2} c\,\Omega_e/\omega_{pe}$. Therefore, unstable MX waves with very large wave numbers, i.e. the most unstable ones, are predominantly transverse waves, with the electric field nearly parallel to the direction of streaming. This is as expected since it is the field component along the streaming motion which causes instability. In fact the instability occurs since kinetic energy associated with the ordered streaming motion of electrons is fed into waves in the form of electromagnetic energy, at a rate proportional to $(u \cdot E)$. Since the maximum amount of energy is fed into waves whose electric field is along the direction of streaming, it results that the most unstable waves are the transverse ones. We finally note that the instability criterion (14) and the maximum growth rate (15) are identical to those of the modified ordinary (MO) mode[1] and, therefore, the MX and MO instabilities are similar for very large wave numbers, caused respectively by streaming motion across and along an external magnetic field.

## IV. STABLE MX WAVES

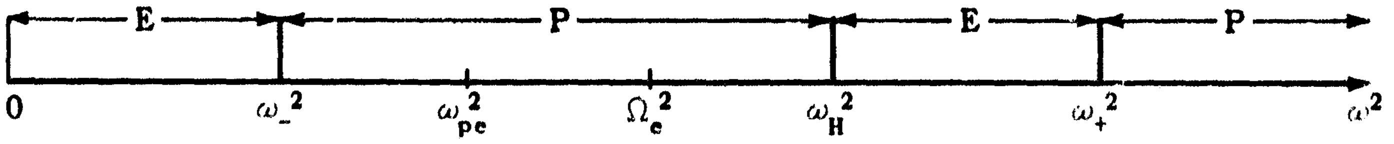
In the limit of no streaming, $u \to 0$, Equation (5) yields the usual dispersion relation for X waves[5]

$$k^2 c^2 = \frac{\left(\omega^2 - \omega_-^2\right)\left(\omega^2 - \omega_+^2\right)}{\omega^2 - \omega_H^2}. \tag{18}$$

The ranges of allowable frequencies are determined by requiring that for real wave numbers propagation occurs only when $k^2 c^2 > 0$. As shown in Figure 1, two propagation bands exist for the electronic extraordinary mode, with a low-frequency and a high-frequency cutoff at $\omega_-$ and $\omega_+$ respectively, and a resonance at $\omega_H$. When the plasma density is sufficiently high and such that $\omega_{pe} > 2^{1/2} |\Omega_e|$, no propagation occurs at frequencies less than the electron cyclotron frequency.

In the presence of a streaming motion, when the dispersion relation is given by Equation (5), the propagation properties of X waves are modified due to the fact that a new cutoff and a new resonance appear, at $|\Omega_e|$ and $\left[\Omega_e^2 - u^2 \omega_{pe}^2/c^2\right]^{1/2}$ respectively. This can produce a new low frequency propagation band. Three possible cases are examined and shown in Figure 2. When the plasma density is sufficiently low and such that $\omega_{pe} < 2^{1/2} |\Omega_e|$, it is possible for $\omega_-$ to be smaller, Case 1, or greater, Case 2, than $\left[\Omega_e^2 - u^2 \omega_{pe}^2/c^2\right]^{1/2}$. In the first case

1. $\omega_{pe}^2 < 2\Omega_e^2$




2. $\omega_{pe}^2 > 2\Omega_e^2$

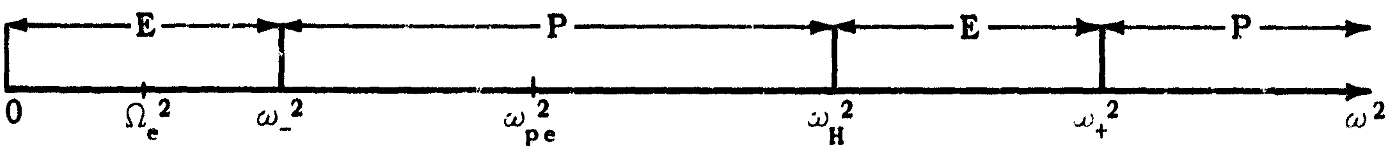
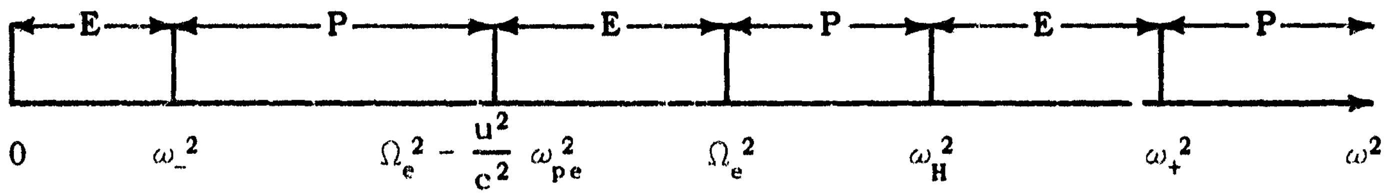


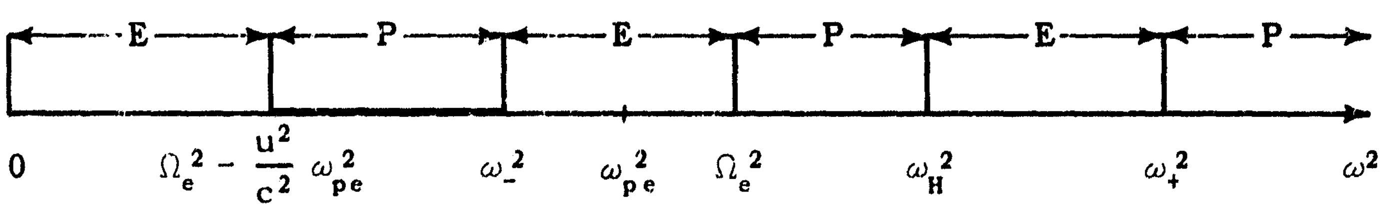

Figure 1. Propagation (P) and evanescent (E) bands of the electronic extraordinary mode.

the low-frequency cutoff is $\omega_-$ (as in the absence of streaming) and there is propagation for $\omega_-^2 < \omega^2 < \left(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right)$, $\Omega_e^2 < \omega^2 < \omega_H^2$ and $\omega^2 > \omega_+^2$. In Case 2, a new low-frequency cutoff appears at $\left[\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right]^{1/2}$ and the allowable frequency ranges are $\left(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right) < \omega^2 < \omega_-^2$, $\Omega_e^2 < \omega^2 < \omega_H^2$ and $\omega^2 > \omega_+^2$. It has to be noted that $\left(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right)$ is greater than zero since $\omega_{pe} < 2^{1/2}\,|\Omega_e|$ and a non-relativistic analysis has been used. When the plasma density is sufficiently high and such that $\omega_{pe} > 2^{1/2}\,|\Omega_e|$, a new low frequency propagation band, $\left(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right) < \omega^2 < \Omega_e^2$, exists for the MX mode. As shown in Figure 2, Case 3, propagation is possible also for $\omega_-^2 < \omega^2 < \omega_H^2$ and $\omega^2 > \omega_+^2$, which are the two propagation bands characteristic of the case in which there is no streaming. The interesting feature of Case 3 is that MX waves can propagate at frequencies below the electron cyclotron frequency and, furthermore, according to this model, for $u/c = |\Omega_e|/\omega_{pe}$ no low-frequency cutoff exists. It should have to be kept in mind, however, that at very low frequencies, it is no longer permissible to disregard the ions and this model becomes invalid since it takes into account only the electron motion. The ion effects will be considered in detail in Section V. For sufficiently high values of the streaming velocity, $\left(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2\right)$ becomes negative and an instability of MX waves with purely imaginary frequencies occurs.

1. $\omega_{pe}^2 < 2\Omega_e^2$; $\quad \frac{u^2}{c^2} < \frac{\Omega_e^2}{2\omega_{pe}^2}\left\{1 + \left(1 + \frac{4\omega_{pe}^2}{\Omega_e^2}\right)^{1/2}\right\} - 1$




2. $\omega_{pe}^2 < 2\Omega_e^2$; $\quad \frac{\Omega_e^2}{2\omega_{pe}^2}\left\{1 + \left(1 + \frac{4\omega_{pe}^2}{\Omega_e^2}\right)^{1/2}\right\} - 1 < \frac{u^2}{c^2} < \frac{\Omega_e^2}{\omega_{pe}^2}$




3. $\omega_{pe}^2 > 2\Omega_e^2$; $\quad \frac{u^2}{c^2} < \frac{\Omega_e^2}{\omega_{pe}^2}$

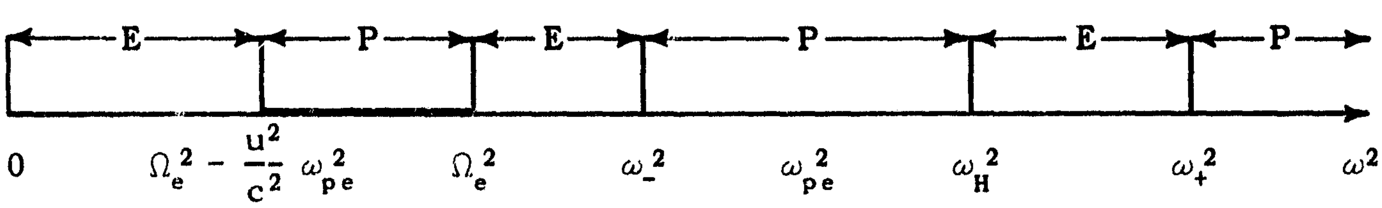



Figure 2. Propagation and evanescent bands of the stable modified extraordinary mode (infinite ion mass). In Case 1, $\omega_{pe}^2$ can be either greater or less than $(\Omega_e^2 - u^2\,\omega_{pe}^2/c^2)$. The new propagation bands $(\Omega_e^2 - u^2/c^2\,\omega_{pe}^2) < \omega^2 < \omega_-^2$ and $(\Omega_e^2 - u^2/c^2\,\omega_{pe}^2) < \omega^2 < \Omega_e^2$ appear in Case 2 and 3 respectively.

## V. COUNTERSTREAMING ELECTRON-ION PLASMAS

So far the ion mass has been assumed infinite so that only the electron dynamics needed to be considered. We are going now to consider the case of two identical counterstreaming electron-ion plasmas, in the approximation in which terms of order $m_e/m_i$ (the ratio of electron and ion mass) are disregarded compared with one. By allowing the ions also to take part in the streaming motion in the system of two identical counterstreaming plasmas described in Section II, the dispersion relation for the MX mode, which follows from Equations (2) and (4), can be written in the following form:

$$k^2 c^2 = \frac{n_x^2 \omega^2 \left(\omega^2 - \Omega_e^2\right) \left(\omega^2 - \Omega_i^2\right)}{\omega^4 - \alpha\omega^2 + \beta} , \tag{19}$$

where $n_x^2$ is the square of the index of refraction for extraordinary waves and is defined as

$$n_x^2 \equiv 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2 \Omega_e^2 \left(\omega^2 - \Omega_i^2 - \omega_{pi}^2\right)}{\omega^2 \left[\omega^4 - \omega^2 \left(\Omega_e^2 + \omega_{pe}^2\right) + \Omega_e^2 \left(\Omega_i^2 + \omega_{pi}^2\right)\right]} ,$$

or

$$n_x^2 = \frac{\left(\omega^2 - \omega_-^2\right) \left(\omega^2 - \omega_+^2\right)}{\left(\omega^2 - \omega_H^2\right) \left(\omega^2 - \omega_{\infty 1}^2\right)} , \tag{20}$$

where $\omega_H$ is the upper hybrid frequency; $\omega_\pm$ are defined by Equations (6) and

$$\omega_{\infty 1}^2 \equiv \frac{\Omega_i^2 + \omega_{pi}^2}{1 + \omega_{pe}^2/\Omega_e^2} . \tag{21}$$

The quantities $\alpha$ and $\beta$ in Equation (19) are given by

$$\alpha \equiv \Omega_e^2 \left(1 - u^2 \omega_{pe}^2/c^2 \Omega_e^2\right) , \qquad \beta \equiv \Omega_e^2 \Omega_i^2 \left(1 - u^2 \omega_{pi}^2/c^2 \Omega_i^2\right) .$$

In Equation (19) the denominator can be rewritten in the form

$$\omega^4 - \alpha\omega^2 + \beta = \left(\omega^2 - \omega_{\infty 2}^2\right)\left(\omega^2 - \omega_{\infty 3}^2\right), \tag{22}$$

where $\omega_{\infty 2}^2$ and $\omega_{\infty 3}^2$ can be approximated as

$$\omega_{\infty 2}^2 \simeq \frac{\Omega_i^2\left(1 - u^2\,\omega_{pi}^2/c^2\,\Omega_i^2\right)}{1 - u^2\,\omega_{pe}^2/c^2\,\Omega_e^2}, \qquad \omega_{\infty 3}^2 \simeq \Omega_e^2\left(1 - u^2\,\omega_{pe}^2/c^2\,\Omega_e^2\right), \tag{23}$$

when

$$\frac{u^2}{c^2} << \frac{\Omega_e^2}{\omega_{pe}^2}\left[1 - 2\left(\frac{m_e}{m_i}\right)^{1/2}\right], \qquad \text{or,} \qquad \frac{u^2}{c^2} >> \frac{\Omega_e^2}{\omega_{pe}^2}\left[1 + 2\left(\frac{m_e}{m_i}\right)^{1/2}\right]. \tag{24}$$

By using Equations (20) and (22), the dispersion relation (19) takes on the form

$$k^2 c^2 = \frac{\omega^2\left(\omega^2 - \Omega_e^2\right)\left(\omega^2 - \Omega_i^2\right)\left(\omega^2 - \omega_-^2\right)\left(\omega^2 - \omega_+^2\right)}{\left(\omega^2 - \omega_H^2\right)\left(\omega^2 - \omega_{\infty 1}^2\right)\left(\omega^2 - \omega_{\infty 2}^2\right)\left(\omega^2 - \omega_{\infty 3}^2\right)}. \tag{25}$$

In the limit of infinite ion mass Equation (25) reduces to Equation (5).* It is possible to establish an instability criterion directly from Equation (25) by looking for purely imaginary roots, i.e. $\omega = \pm i\gamma$, which correspond to a pair of growing and decaying waves. In such a case, from the requirement that the first member of Equation (25) be positive for real wave numbers and from conditions (24), it results that the MX mode is unstable, for any wave number, if

$$\frac{\Omega_i}{\omega_{pi}} < \frac{u}{c} << \frac{\Omega_e}{\omega_{pe}}\left[1 - \left(\frac{m_e}{m_i}\right)^{1/2}\right], \tag{27}$$

*Since Equation (25) is a fifth degree algebraic equation in $\omega^2$, it is difficult to obtain analytically a sufficient condition for stability, i.e. the case in which all roots for $\omega^2$ are real and positive.

with maximum growth rate

$$\gamma_i = \frac{\left[\dfrac{u^2 \omega_{pi}}{c^2} - \Omega_i^2\right]^{1/2}}{\left[1 - \dfrac{u^2 \omega_{pe}^2}{c^2 \Omega_e^2}\right]^{1/2}} \;; \tag{28}$$

and

$$\frac{u}{c} >> \frac{\Omega_e}{\omega_{pe}} \left[1 + \left(\frac{m_e}{m_i}\right)^{1/2}\right], \tag{29}$$

with maximum growth rate

$$\gamma_e = \left[\frac{u^2 \omega_{pe}^2}{c^2} - \Omega_e^2\right]^{1/2} , \tag{30}$$

which is equal to the maximum growth rate (15) for the case in which the ions are disregarded.

The research of K. F. Lee is supported by the National Science Foundation under grant GK-4592.

REFERENCES AND FOOTNOTES

1. K. F. Lee, Phys. Rev. Letters, 21, 1439 (1968); Phys. Rev. 181, 447, 453 (1969).

2. B. B. Kadomtsev, Plasma Physics and the Problem of Controlled Thermonuclear Reactions, Acad. Sci. U.S.S.R., 1958, p. 431.

3. D. W. Ross, Report Center for Plasma Physics and Thermonuclear Research, Austin, Texas, No. CPPT17, ORO-3458-19, 1969.

4. If the relative streaming motion is along instead of across the magnetic field, the ordinary mode is modified, while the extraordinary mode is unaffected, for a system of two identical counterstreaming plasmas, Refs. 1-3.

5. Handbook of Chemistry and Physics (Chemical Rubber Publishing Co., Cleveland, Ohio, 1965) 46th ed., p. A-189.

6. In the limit of no streaming, when the stable MX waves reduce to the usual extraordinary waves, it is $|E_x| \gg |E_y|$, i.e., the waves with an infinitely great refractive index are longitudinal.

7. K. G. Budden, Radio Waves in the Ionosphere (Cambridge University Press, New York, 1961) p. 61.